

The AdS/CFT Correspondence in Two Dimensions

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We review recent progress in understanding the anti-de Sitter/conformal field theory correspondence in the context of two-dimensional (2D) dilaton gravity theory.

I. INTRODUCTION

The $d = 2$ case [1,2] of the correspondence between gravity on anti-de Sitter (AdS) space and conformal field theory (CFT) [3] is important for several reasons. First of all, the CFT involved has an infinite dimensional symmetry, so that the theory is highly constrained. Analogously to the $d = 3$ case [4], gravitational structures (e.g. black holes) can be investigated using conformal field theory techniques. Secondly, AdS₂ appears as near-horizon geometry of a variety of higher dimensional black holes not only in string theory but also in the general relativity context (the Reissner-Nordstrom solution). Last but not least, being the simplest case of the correspondence, the AdS₂/CFT₁ duality can be used to test general ideas about the correspondence in particular and the holographic principle in general. Of particular conceptual relevance is the fact that it should provide a correspondence between a field theory (2D gravity) and conformal mechanics.

Contradicting the general belief that low-dimensional physics is simpler than the higher-dimensional one, the AdS₂/CFT₁ duality has many puzzling and controversial features. These puzzling features are related with the peculiarities of 2D gravity. In two dimensions there are at least three (related and unrelated) versions of gravity theories: dilaton gravity, random surfaces and string theory with 2D target space (more generally, one can consider string compactifications on ten-dimensional backgrounds containing AdS₂, e.g. AdS₂ × S² × T⁶). Perhaps the most striking peculiarity of the 2D case is the topology of the AdS space involved. Full AdS spacetime in $d = 2$ has cylindrical topology, so that its boundary is not connected, making difficult the identification of the boundary CFT that should be dual to the gravity theory. Owing to this difficulties it is almost impossible to discuss the correspondence in general. We will focus on 2D dilaton gravity.

II. ASYMPTOTIC SYMMETRIES OF ADS₂ AND THE CONFORMAL GROUP

Two-dimensional spacetimes with constant negative curvature, $R = -2\lambda^2$ (in the following they will be referred to as AdS₂) appear as dynamical solutions of dilatonic gravity in two dimensions [5]. The simplest case is represented by the Jackiw-Teitelboim (JT) model [6],

$$A = \frac{1}{2} \int \sqrt{-g} d^2x \Phi (R + 2\lambda^2), \quad (1)$$

where Φ is a scalar field related to the usual definition of the dilaton ϕ by $\Phi = \exp(-2\phi)$. The model admits black hole solutions, which have the form [5]

$$ds^2 = -(\lambda^2 r^2 - \frac{2m_{bh}}{\lambda\Phi_0}) dt^2 + (\lambda^2 r^2 - \frac{2m_{bh}}{\lambda\Phi_0})^{-1} dr^2, \\ \Phi = \Phi_0 \lambda r. \quad (2)$$

An important property of these black hole solutions is that they are locally equivalent, modulo 2D diffeomorphisms, to the $m_{bh} = 0$ vacuum solution. Moreover, the global feature of the spacetime are such that the vacuum solution has to be considered as a portion of full AdS₂ (which is a geodetically complete spacetime). The vacuum has a null boundary at $r = 0$. In this way we avoid the difficulty related with the cylindrical topology of AdS₂: our reference spacetime has only one timelike boundary at $r \rightarrow \infty$. But we pay the price of having a singular geodetically incomplete spacetime, because now it has a "singularity" at $r = 0$.

The black hole solution (2) can be interpreted as a thermodynamic system. The black hole mass depends quadratically on both the Hawking temperature T and the entropy S_{bh} [5],

$$m_{bh} = \frac{2\pi^2\Phi_0}{\lambda} T^2, \quad S_{bh} = 4\pi\sqrt{\frac{m\Phi_0}{2\lambda}}. \quad (3)$$

AdS₂ is a maximally symmetric space; it admits, therefore, three Killing vectors generating the $SO(1,2) \sim SL(2,R)$ group of isometries. The asymptotic symmetries of AdS₂ are by definition the subgroup of the 2D diffeomorphisms group that leaves invariant the leading term in the $r \rightarrow \infty$ asymptotical expansion of the metric tensor, i.e. they preserve the large r behavior

$$\begin{aligned}
g_{tt} &= -\lambda^2 r^2 + \gamma_{tt}(t) + \mathcal{O}\left(\frac{1}{r^2}\right), \\
g_{tr} &= \frac{\gamma_{tr}(t)}{\lambda^3 r^3} + \mathcal{O}\left(\frac{1}{r^5}\right), \\
g_{rr} &= \frac{1}{\lambda^2 r^2} + \frac{\gamma_{rr}(t)}{\lambda^4 r^4} + \mathcal{O}\left(\frac{1}{r^6}\right), \\
\Phi &= \Phi_0 \left(\lambda \rho(t) r + \frac{\gamma_\Phi(t)}{2\lambda r} \right) + \mathcal{O}\left(\frac{1}{r^3}\right), \quad (4)
\end{aligned}$$

where the fields $\gamma_{\mu\nu}, \gamma_\Phi, \rho$ parametrize the first sub-leading terms in the expansion and can be interpreted as deformations of the boundary of AdS_2 and of the dilaton.

The form of the boundary conditions for the dilaton Φ are determined by requiring consistency with the action of the diff_2 group. They require a (asymptotically) nonconstant dilaton, which breaks the $SL(2, R)$ group of isometries (and in general the asymptotical symmetries group (ASG) of the metric) of AdS_2 . This symmetry breaking is related with the appearance of a central charge in the related conformal algebra [7]. The asymptotic form (4) is preserved by infinitesimal diffeomorphisms $\chi^\mu(x, t)$ of the form [2],

$$\begin{aligned}
\chi^t &= \epsilon(t) + \frac{\ddot{\epsilon}(t)}{2\lambda^4 r^2} + \frac{\alpha^t(t)}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right), \\
\chi^r &= -r\dot{\epsilon}(t) + \frac{\alpha^r(t)}{r} + \mathcal{O}\left(\frac{1}{r^2}\right). \quad (5)
\end{aligned}$$

Expanding $\epsilon(t)$ in series one finds that the generators of the symmetry, L_k , satisfy a Virasoro algebra $[L_k, L_l] = (k-l)L_{k+l}$. The ASG of AdS_2 can be, therefore, identified with the one-dimensional conformal group acting on the boundary of AdS_2 . Moreover, the boundary fields appearing in Eq. (4) span a representation of this group. Under the action of the group they transform as conformal fields with definite conformal dimensions [7].

The Virasoro algebra associated with the ASG can be centrally extended. The value of the central charge C of the algebra is particularly important because enables one to give a statistical interpretation of the thermodynamical behavior (3) of the black hole. For a generic CFT we have the energy-temperature and entropy-mass relations [8],

$$m_{CFT} = \frac{\pi}{12} \alpha' C T^2, \quad S_{CFT} = 2\pi \sqrt{\frac{C m_{CFT}}{6}}. \quad (6)$$

The central charge of the algebra can be computed using a canonical realization of the ASG [2,7]. The asymptotic symmetries define charges J , which in the canonical formalism give a realization of the ASG, through the Dirac brackets

$$\{J[\chi], J[\omega]\}_{DB} = J[[\chi, \omega]] + C(\chi, \omega). \quad (7)$$

Using the associated deformation algebra one can calculate the central charge $C(\chi, \omega)$. The central charge was first calculated in Ref. [2]. The value $C = 24\Phi_0$ was found, which turned out to be wrong by a factor of 2 [2,7] (see also Ref. [9]). The puzzle was resolved independently in Refs [10], [11]. In particular in Ref. [10], it was shown that because of the inner boundary of the spacetime at $r = 0$, the central charge has an additional contribution C_{ent} due to the entanglement of states. C_{ent} can be calculated using the coordinate transformation, which maps the vacuum $m_{bh} = 0$ into the black hole solution together, together with the anomalous transformation law of the energy momentum tensor T_{tt} . One finds $C_{ent} = -12\Phi_0$, so that $C_{tot} = C + C_{ent} = 12\Phi_0$. Inserting this value of the central charge into the relations (6), one reproduces exactly the thermodynamical relations (3).

III. THE SIGMA MODEL APPROACH

Two-dimensional dilaton gravity can be formulated as a nonlinear sigma model [12]. For the JT model the action reads

$$A = \frac{1}{2} \int_{\Sigma} d^2x \sqrt{-g} \frac{\partial_\mu \phi \partial^\mu M}{\Phi^2 - M}, \quad (8)$$

where M is the mass functional [13], which on the classical orbit is constant and proportional to the ADM mass of the black hole, $M = 2\Phi_0 \lambda m_{bh}$. Expanding near $\Psi = (-2\lambda^2 \Phi)^{-1} = 0$ one has [14]

$$A = \int d^2x \sqrt{-g} \partial_\mu M \partial^\mu \psi \left[1 + \sum_{k=1}^{+\infty} (2\lambda)^{2k} M^k \psi^{2k} \right]. \quad (9)$$

Keeping in mind that Φ^{-1} is the (coordinate dependent) coupling constant of the gravitational theory, one easily realizes that equation (9) is both a weak-coupling and a near-boundary (around $r = \infty$) expansion. The leading term describes a free CFT_2 , which by means of a trivial field redefinition can be cast in the form of a bosonic string with 2D target-space [14],

$$A_0 = \frac{1}{2\pi\alpha'} \int d^2z \partial X^\mu \bar{\partial} X_\mu. \quad (10)$$

Since AdS_2 has a timelike boundary the action (10) must necessarily describe open strings. On the boundary we can impose either Dirichlet [$\partial_a X^\mu(x=0) = 0$] or Neumann [$n^a \partial_a X^\mu(x=0) = 0$, where n^a is the normal to the boundary] boundary conditions.

Because the weak-coupling expansion (9) is also a near-boundary expansion, one expects the symmetry group of the open string to be related with the ASG of AdS_2 and gravitational asymptotical modes to have an interpretation in terms of string normal modes. One can show that the symmetry group of the string can be obtained

from the Killing vectors (5), generating the ASG, by simply fixing the higher order terms in the expansion the Killing vectors (5) (the so-called pure gauge diffeomorphisms). Moreover the boundary fields γ, ρ of Eq. (4) have a natural interpretation in terms of CFT₂ fields. They transform in the holomorphic sector of the open string theory as fields of a given conformal dimension.

We can also write down explicitly the relationship between gravitational modes $M_{m,n}, \Psi_{m,n}$ appearing in the asymptotical expansion of the fields M, Ψ and string normal modes α_m^μ . We have [14]

$$\begin{aligned}\alpha_m^\mu &= -i\sqrt{\pi}2^{-1/2-m} (mM_{0,-m}), \\ \alpha_m^\mu &= i\sqrt{\pi}2^{-1/2-m} [M_{1,-1-m} \mp \Psi_{1,-1-m}],\end{aligned}\quad (11)$$

respectively for Neumann and Dirichlet boundary conditions.

A crucial difference between the two sets of boundary conditions is that, whereas in the Dirichlet case the Virasoro operators L_k can be written in terms of local string oscillators, in the Neumann case using Eq.(11) we find $L_k = 0$ identically. This means that only in the case of Dirichlet boundary conditions the ASG can be realized using local string oscillators. Neumann boundary conditions correspond to the realization of the symmetry given by the charges J , which has been described in the previous section.

The main lesson following from the nonlinear sigma model approach can be summarized as follows. AdS₂ dilaton gravity has two $\Phi \rightarrow \infty$ degeneration limits. The first defines a duality with an open string with Dirichlet boundary conditions, can be realized using local string oscillators and describes a AdS₂/CFT₂ correspondence. The second defines a duality with an open string with Neumann boundary conditions, cannot be realized using local string oscillators and describes a AdS₂/CFT₁ correspondence. In the next section we will identify unambiguously the CFT₁ involved in the latter correspondence as a conformal mechanics living in the $r = \infty$ boundary of AdS₂.

Also in the Dirichlet case we can use the AdS/CFT duality to describe the 2D black hole (5) as a CFT object. In particular, we can reproduce the thermodynamical entropy by counting the degeneracy of states in the CFT. Computing the central charge of the Virasoro algebra using its interpretation as Casimir energy, one finds the same result obtained in the previous section, $C = 12\Phi_0$ [14], which, in turn, inserted in Eq. (6) reproduces the Thermodynamical parameters (3) of the black hole.

IV. ADS₂ GRAVITY AND CONFORMAL MECHANICS

The previous sections gave us a strong hint about the existence of a conformal mechanics description of the

weak-coupling regime of AdS₂ gravity. Let us now identify unambiguously this conformal mechanics. Using the boundary expansion (4) in the field equations for AdS₂ dilaton gravity following from the action (1) and taking the $r \rightarrow \infty$ limit, one gets the dynamics induced by the bulk gravity theory on the boundary [10],

$$\begin{aligned}\lambda^{-2}\ddot{\rho} - \rho\gamma + \beta &= 0, \\ \dot{\rho}\gamma + \dot{\beta} &= 0,\end{aligned}\quad (12)$$

where $\beta = \frac{1}{2}\rho\gamma_{rr} + \gamma_\Phi$ and $\gamma = \gamma_{tt} - \frac{1}{2}\gamma_{rr}$. These equations of motion define a conformal mechanics. In fact they are invariant under the diff₁ group $\delta(t) = \epsilon(t)$ realized as [10],

$$\begin{aligned}\delta\rho &= \epsilon\dot{\rho} - \dot{\epsilon}\rho, \\ \delta\beta &= \epsilon\dot{\beta} + \dot{\epsilon}\beta + \frac{\ddot{\epsilon}\rho}{\lambda^2}, \\ \delta\gamma &= \epsilon\dot{\gamma} + 2\dot{\epsilon}\gamma - \frac{\ddot{\epsilon}}{\lambda^2}.\end{aligned}\quad (13)$$

The equation of motions (12) describe a mechanical system with anholonomic constraints. Alternatively, introducing the new coordinate $q = \sqrt{\rho/\lambda}$ with conformal dimension $-1/2$, one can show that Eq. (12) are equivalent to the equation [10],

$$\ddot{q} - \frac{g}{q^3} = \frac{\lambda^2}{2}\gamma q, \quad (14)$$

together with the Hamiltonian constraint

$$\frac{\dot{q}^2}{2} + \frac{g}{2q^2} = -\frac{\lambda^2}{4}\beta. \quad (15)$$

The equation of motion (14) describes the (IR-regularized) De-Alfaro-Fubini-Furlan (DFF) conformal mechanics [15] coupled with an external source γ . It can be derived from the action [10]

$$I = \int dt \left[\frac{1}{2}\dot{q}^2 - \frac{g}{2q^2} + \frac{1}{4}\lambda^2\gamma q^2 \right]. \quad (16)$$

This action is invariant under the full conformal group (diff₁). We can therefore interpret the theory as the CFT₁ describing the weak-coupling regime of AdS₂ gravity. In analogy with CFT₂ we can write down the associated energy-momentum tensor [10]

$$T_{tt} = \lambda^2(\dot{\rho}\gamma + \dot{\beta} + M) - 2\Phi_0\ddot{\rho}. \quad (17)$$

We can think of γ as a external source or, alternatively, as the time-dependent coupling of the harmonic oscillator potential term $\sim \gamma q^2$. Because γ is arbitrary and time-dependent, the mechanical system is nondeterministic and the energy is not conserved, as it is evident from the Hamiltonian (15). We have here a strong analogy with disordered systems in statistical mechanics. From

the point of view of the 2D gravity theory γ encodes the information about the gauge symmetry of the gravity theory. These facts give a strong hint on the existence of a deep relationship between gauge symmetries of fields theories and nondeterministic dynamical systems.

V. A MORE GENERAL CORRESPONDENCE?

The discussion of the previous sections has been focalized on the particular 2D dilaton gravity model (the JT theory) for which the curvature of the spacetime is everywhere constant. However, the main results of our approach can be easily generalized to models admitting solutions that are AdS_2 *only* asymptotically [2]. The generalization to the other formulations of 2D gravity mentioned in the introduction (string theory on backgrounds containing AdS_2 and random surfaces) is more involved.

The AdS_2 gravity theory we have discussed in this presentation (or generalization of it) can be considered as the “effective theory” of the two formulation of 2D gravity mentioned above. In the first case as the effective theory originating from $10\text{D} \rightarrow 2\text{D}$ string compactifications. In the second case (random surfaces) as some sort of Liouville theory describing the dynamics of the conformal mode (now identified with the dilaton) (See also Ref. [16]).

The main problem of this schema is that our approach works only for string compactifications that produce AdS_2 endowed with a *non constant* dilaton [17]. We are not able to describe the interesting class of compactifications characterized by a *constant* dilaton. These solutions, whose prototype is the Bertotti-Robinson solution of general relativity, have a thermodynamical behavior that differs drastically from that described here. They are characterized by a $T = 0$ degenerate ground state, separated by a mass gap from the continuous part of the spectrum. The form of the AdS/CFT correspondence for this class of models is still an open question.

On the other hand the correspondence we have found between AdS_2 gravity and DFF conformal mechanics coupled with an external source indicates that the $\text{AdS}_2/\text{CFT}_1$ correspondence could be a particular case of a more general correspondence between boundary CFT’s and large N limit of mechanical system. Progress along this direction has been achieved in Ref. [18], and Ref. [19]. In Ref. [18] it has been shown that the physical spectrum and physical states of a bosonic string with 2D target space can be put in correspondence with the large N limit a mechanical system defined essentially by decoupled harmonic oscillators. This mechanical system has a natural interpretation in terms of a one-dimensional stochastic process. In the second paper [19], it has been shown that the large N limit of Calogero models is equivalent to a CFT.

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